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# SOME OBSERVATIONS ON FACTOR ANALYSIS

Harry H. Harman

While my primary interests in factor analysis have been concerned with the development of its methodology, I wish, at this time, to reminisce somewhat about the origin and growth of the subject, and to make an appraisal of its present status. This paper, therefore, will be non-mathematical, expository in character.

The birth of factor analysis is generally ascribed to Charles Spearman. His monumental work in developing a psychological theory involving a single general factor and a number of specific factors goes back to 1904 when his paper, "General Intelligence, Objectively Determined and Measured", was published in the American Journal of Psychology. Of course, his 1904 investigation was only the beginning of his work in developing the Two-Factor Theory, and this early work is not "explicitly in terms of factors". Perhaps a more crucial article, certainly insofar as the statistical aspects are concerned, is the 1901 paper by Karl Pearson in which he sets forth "the method of principal axes". Nevertheless, Spearman, who devoted the remaining 40 years of his life to the development of factor analysis, is regarded as the father of the subject.

A considerable amount of work on the psychological theories and mathematical foundations of factor analysis followed in the next twenty years. The principal contributors during this period included Charles Spearman, Cyril Burt, Karl Pearson, Godfrey H. Thomson, J. C. Maxwell Garnett, and Karl Holzinger; and the topics receiving the greatest attention were concerned with the proof or disproof of the existence of general ability, the study of sampling errors of tetrad differences, and computational methods for a single

general factor which included the fundamental formula of the centroid solution.

The early modern period including the bulk of the active and published controversy on factor analysis, came after 1925, with a real spurt of activity in the 1930's. By this time it had become quite apparent that Spearman's Two-Factor Theory was not always adequate to describe a battery of tests. So group factors found their way into factor analysis; although the experimenters, at first, were very reluctant to admit such deviation from the basic theory and restricted the group factors to as small a number as possible. What actually happened was that the theory of a general and specific factors in Spearman's original form was superseded by theories of many group factors, but the early method continued to be employed to determine these many factors. Then it naturally followed that some workers explored the possibility of extracting several factors directly from a matrix of correlations among tests, and thus arose the concept of multiple-factor analysis (Garnett, 1919).

While the actual term may be due to Thurstone, and while he undoubtedly has done most to popularise the method of multiple-factor analysis, he certainly was not the first to take exception with Spearman's Two-Factor Theory and was not the first to develop a theory of many factors. It is not even the centroid method of analysis for which Thurstone deserves a place of prominence in factor analysis. The centroid method is clearly admitted by Thurstone to be a computational compromise for the principal-factor solution. The truly remarkable contribution of Thurstone was the generalisation of Spearman's tetrad-difference criterion to the rank of the correlation matrix as the basis for determination of the number of common factors. He saw that a zero tetrad-difference corresponded to the vanishing of a second-order determinant, and extended this notion to the vanishing of higher order

determinants as the condition for more than a single factor. The matrix formulation of the problem has greatly facilitated further advances in factor analysis.

Let us turn now from the review of the historical developments, to consider how the several schools of modern factor analysis arose in psychology. As is well known, a given matrix of correlations can be factored in an infinite number of different ways. (It is not entirely clear whether this well known fact was truly appreciated in the earlier days of factor analysis; and if, in fact, the failure to recognise this mathematical truism may not have been the cause of the many controversies regarding the "true", the "best", or the "invariant" solution for a set of data.) When an infinite number of equally accurate solutions are available, the question arises: How shall a choice be made among these possibilities? The preferred types of factor solutions are determined on the basis of two general principles: (1) statistical simplicity, and (2) psychological meaningfulness. In turn, each of these principles requires interpretation, and each has been applied variously to yield several distinct schools of factor analysts.

If one were to make his choice entirely upon statistical considerations, a rather natural approach would be to represent the original set of variables in terms of a number of factors, determined in sequence so that at each successive stage the factor would account for a maximum of the variance. This statistically optimal solution — the method of principal axes — was first proposed by Pearson at the turn of the century, and in the 1930's, Hotelling provided the full development of the method, including an iterative process for the determination of the characteristic roots. While this procedure is perfectly straightforward, it entails a very considerable amount of computation, and becomes impractical with ordinary computing facilities

when the matrix is of order 10 or greater. In recent years, however, this difficulty has been overcome by the use of high speed electronic computers.

Another choice based upon statistical considerations is the Centroid Solution. As indicated above, this method was introduced only as a computational expedient when it became apparent that the principal factor solution was too laborious. All that can be said for the centroid method is that it produces without much arithmetic one of many possible sets of axes which account for the variance in a manner approximating the optimal situation of the principal axes.

The end product of these solutions — principal or centroid — is not acceptable to psychologists (with the possible exception that the principal factor solution is sometimes acceptable to Burt). In quest of "meaningful" factor solutions, psychologists have introduced various theories in the hope of arriving at a form of solution which would be unique and apply equally well to intelligence, personality, physical measurements, and any other variables with which they might be concerned. Holzinger's Bi-Factor Theory and Thurstone's Simple Structure Theory are in this class. On the other hand, Thomson's Sampling Theory is primarily a psychological theory of the mind. There is no preferred type of factor solution obtainable uniquely on grounds of psychological significance. If psychological meaningfulness rather than a pure statistical standard is imposed, then to some extent the judgment of the investigator will be involved. Attempts at an objective solution to this problem will be indicated in a moment. However, I made no attempt to cover the work of J. P. Guilford, Raymond B. Cattell, John French, and others concerned with the study and isolation of specific psychological factors.

First, I wish to elaborate on the question of indeterminacy in factor analysis. The infinitude of factorisations of a correlation matrix may

perhaps better be comprehended from a geometric interpretation of the situation. We can regard the observations on a set of variables as determining a number of vectors (corresponding to the variables) in a space equal to the number of subjects. By methods of factor analysis these vectors can generally be contained in a space of smaller dimension than the number of variables. The coordinate axes of this reduced space are the common factors, and the original variables can be expressed linearly in terms of these factors. The determination of this common-factor space is in no way dependent on the particular coordinate frame of reference employed. This arbitrariness is represented geometrically by the infinite number of rotations possible from one set of coordinate axes to another.

For ease of mathematical description, and sometimes to facilitate psychological interpretation, it is common practice to change the frame of reference. In making such a transformation of coordinates it must be remembered that the geometric configuration, e.g., straight line or swarm of points, is left unaltered. The mathematical expression or formulas describing the geometric configuration may change under transformation, but the configuration itself is invariant.

The <u>mathematician</u> usually is concerned with the geometric configuration only, using the frame of reference as a tool, and will prefer one reference system to another if it yields a simpler (and more elegant) expression for his configuration. For example, the elaborate formula consisting of six terms:

$$AX^2 + BY^2 + CXY + DX + EY + F = 0$$

represents a geometric configuration (an ellipse) in one (arbitrary) frame of reference while the expression

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$

represents precisely the same configuration in another (arbitrary) reference frame selected so as to make the equation as simple as possible.

Unlike the mathematician, the psychologist frequently concerns himself with the interpretation of the frame of reference, using the configuration of points merely as the vehicle to get to the particular reference axes. Thus, in factor analysis the geometric configuration is a swarm of points, each one representing a test and the density of the points being a function of the intercorrelations among the tests. A frame of reference may be selected for psychological interpretation on the basis of the particular configuration of points but the emphasis in the resulting psychological theory is on the coordinate axes, not the configuration alone.

The attempt to fix the coordinate axes on some objective basis has been undertaken by a number of psychologists. This work was spearheaded by Thurstone's principle of "Simple Structure", and he was among the first to strive for an objective definition of this concept and an accompanying objective procedure for a simple structure solution. Since 1935, Thurstone has been followed by Horst, Tucker, Carroll, Ferguson, Saunders, and Wrigley with specific proposals for analytical or semi-analytical procedures for the attainment of simple structure or approximations to it. Real strides in this direction have been made very recently, and reported in the last two years.

In general, the rotation of axes in order to arrive at simple structure may be viewed as an attempt to reduce the complexity of the factorial description of the tests. The ultimate objective would be a uni-factor solution, in which each test would be of complexity one, i.e., involve only a single common factor. An orthogonal uni-factor solution is extremely

unlikely with empirical data (except for the limiting case of only a general factor for the entire battery of tests). If a uni-factor solution were possible, the variance of each test would result from but one factor loading; and a reasonable approach to this ideal would seem to require the maximum inequality in the distribution of the variance among the several factors. This implies an orthogonal transformation which maximizes the variance of the contributions of factors (i.e., the squared factor loadings). Basically, then, the analytical approach to simple structure involves the maximization of fourth powers of factor loadings. For this reason, Wrigley has named this approach the "Quartimax Method." His method is mathematically equivalent to Carroll's, although the two procedures were developed to satisfy different criteria. Also, Ferguson and Saunders had independently arrived at very similar results. Since a name has been appended to this type of analytical procedure, it is quite likely that the several independent methods will collectively be identified as the "Quartimax Method."

In addition to the problem of obtaining an objective solution which would be psychologically acceptable, factor analysts have been troubled by two major weaknesses — one in theory and the other in practice — being, respectively,

1) Lack of rigorous tests of significance, and
2) Excessive computational requirements.

Lawley's work on the maximum likelihood method during the past 15 years marks the beginning of a period which we hope will eventually lead to a resolution of the first of these deficiencies. The second area of difficulty is being resolved by the advent of high speed electronic computers. Today, such machines as the Ordvac at Aberdeen, the Illiac at the University of Illinois, and the Whirlwind at MIT, have contributed to the solution of

problems in factor analysis in five distinct areas:

1) The rapid computation of correlation coefficients, making the techniques of factor analysis feasible.

2) The actual computation of factor solutions — principal axes or square root solution for very large matrices.

3) The estimation of communalities.

4) Analytic approach to "simple structure."

5) Lawley's maximum likelihood method, and significance tests.

It is safe to assume that high speed digital computers will become increasingly available for factor analysis work. While innumerable advantages will immediately accrue to the scientist, there are also dangers in the greater ease of computations. He will not have such intimate knowledge of his data as in the case of a person who spends considerable time on a desk calculator. The ease with which computations can be accomplished in this electronic age may lead to excesses and wasted effort. However, the ultimate effect of the high speed computers may best be described in Wrigley's words: "...(they) are going to play much the same role for the statistically-minded psychologist as the telescope has for the astronomer, and the microscope for the biologist — the role of an instrument which leads to greatly widened horizons."